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Note on the Projection of the General Locus of space of four dimensions into space of three dimensions.

By Thomas Craig, *Fellow of the Johns Hopkins University.*

In the first number of this Journal, Professor Newcomb has given a proof of a certain property of a closed surface, viz: that such a surface, if placed in space of four dimensions, could be turned inside out without stretching or tearing it. Of course this is but one out of the many new degrees of freedom which material objects would possess if placed in four-dimensional space; but it is not the object of this brief paper to study the new properties of natural objects placed in space of four dimensions, but rather to determine the representations in space of three dimensions of the loci which can only exist in four-dimensional space. The general locus which characterizes space of four dimensions may be represented by an equation between the four rectangular coordinates x, y, z, t. This locus is, in fact, *the general locus of three dimensions.* A *surface* in four-dimensional space requires for its determination two equations between the variables, and, if the surface is one which may exist in Euclidean space, one of these equations must be linear or of the form

$$\alpha x + \beta y + \gamma z + \delta t + \varepsilon = 0,$$

which denotes the infinite plane Euclidean space. There are of course an infinite number of these Euclidean spaces in four-dimensional space which are determined by assigning different values to the constants α, β, &c.

The two equations which taken together denote a surface in four-dimensional space sustain to this surface the same relation that two equations in Euclidean space sustain to the curve which is the intersection of surfaces corresponding to each of these equations. A surface in four-dimensional space is then given by

$$F(x, y, z, t) = 0,$$
$$\Phi(x, y, z, t) = 0,$$

and $F = 0$ and $\Phi = 0$, each denotes the general three-dimensional locus characteristic of space of four dimensions, just as the surface $F(x, y, z) = 0$ is the general two-dimensional locus characteristic of space of three dimensions.

I obtain in the following paper merely the general differential equations which are necessary to obtain the representation in Euclidean space of the locus $F(x, y, z, t) = 0$. A transformation to three independent variable parameters would probably have the effect of simplifying the results, as in Euclidean space the whole theory of surfaces, or more exactly of the curvature of surfaces, is more conveniently and elegantly studied by using two independent variable parameters instead of the three dependent rectangular coordinates of any point.

The locus in three-dimensional space $F(x, y, z) = 0$ is projected into two-dimensional space, say the plane $z' = 0$, by simply finding the values of x', y', the rectangular coordinates of the point in the plane corresponding to the point x, y, z of the surface.

Of course the process of finding these values is dependent upon the conditions which are to be fulfilled in making the projection. The condition of projection that we shall here employ is that of the similarity of the elements of the given locus with the representation of these elements in three-dimensional space. The conditions to be fulfilled then are, the preservation of angles in their true size, and the preservation of the ratio between any two elements of the given locus in the projection of these elements.

The equation of the general four-dimensional locus is

$$F(x, y, z, t) = 0.$$

Let ξ, η, ζ denote the rectangular coordinates in space of three dimensions; it is required to find ξ, η, ζ in terms of x, y, z, t, so that the above conditions shall be fulfilled. Since ξ, η, ζ are functions of x, y, z, t, we have

$$d\xi = \frac{\partial\xi}{\partial x}dx + \frac{\partial\xi}{\partial y}dy + \frac{\partial\xi}{\partial z}dz + \frac{\partial\xi}{\partial t}dt,$$

$$d\eta = \frac{\partial\eta}{\partial x}dx + \frac{\partial\eta}{\partial y}dy + \frac{\partial\eta}{\partial z}dz + \frac{\partial\eta}{\partial t}dt,$$

$$d\zeta = \frac{\partial\zeta}{\partial x}dx + \frac{\partial\zeta}{\partial y}dy + \frac{\partial\zeta}{\partial z}dz + \frac{\partial\zeta}{\partial t}dt,$$

and also

$$0 = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial t}dt.$$

Write for brevity

$$\frac{\partial F}{\partial x} = \alpha, \quad \frac{\partial F}{\partial y} = \beta, \quad \frac{\partial F}{\partial z} = \gamma, \quad \frac{\partial F}{\partial t} = \varepsilon,$$

$$\frac{\partial\xi}{\partial x} = a, \quad \frac{\partial\xi}{\partial y} = b, \quad \frac{\partial\xi}{\partial z} = c, \quad \frac{\partial\xi}{\partial t} = e,$$

$$\frac{\partial\eta}{\partial x} = a', \quad \frac{\partial\eta}{\partial y} = b', \quad \frac{\partial\eta}{\partial z} = c', \quad \frac{\partial\eta}{\partial t} = e',$$

$$\frac{\partial\zeta}{\partial x} = a'', \quad \frac{\partial\zeta}{\partial y} = b'', \quad \frac{\partial\zeta}{\partial z} = c'', \quad \frac{\partial\zeta}{\partial t} = e'';$$

then the preceding equations become

$$\begin{aligned}\alpha dx + \beta dy + \gamma dz + \varepsilon dt &= 0\,,\\ adx + bdy + cdz + edt &= d\xi\,,\\ a'dx + b'dy + c'dz + e'dt &= d\eta\,,\\ a''dx + b''dy + c''dz + e''dt &= d\zeta\,.\end{aligned}$$

We will define here ∇ as the determinant of the above equations when $d\xi$, $d\eta$, $d\zeta = 0$, that is,

$$\nabla = \begin{vmatrix} \alpha, & \beta, & \gamma, & \varepsilon \\ a, & b, & c, & e \\ a', & b', & c', & e' \\ a'', & b'', & c'', & e'' \end{vmatrix}.$$

The minors of ∇ corresponding to α, β, γ, ε are

$$K,\ L,\ M,\ N$$

corresponding to a, b, c, e,

$$k,\ l,\ m,\ n$$

and similarly those corresponding to a', b', &c., a'', b'', &c., are

$$\begin{gathered} k',\ l',\ m',\ n',\\ k'',\ l'',\ m'',\ n''. \end{gathered}$$

If we suppose, now, that in threefold space ζ is the only quantity which varies, we must have $d\xi = d\eta = 0$, and

$$\begin{aligned}\alpha dx + \beta dy + \gamma dz + \varepsilon dt &= 0\,,\\ adx + bdy + cdz + edt &= 0\,,\\ a'dx + b'dy + c'dz + e'dt &= 0\,;\end{aligned}$$

from these follow

$$\frac{dx}{k''} = -\frac{dy}{l''} = \frac{dz}{m''} = -\frac{dt}{n''}.$$

Write

$$\begin{aligned}\Omega &= \sqrt{k^2 + l^2 + m^2 + n^2}\,,\\ \Omega' &= \sqrt{k'^2 + l'^2 + m'^2 + n'^2}\,,\\ \Omega'' &= \sqrt{k''^2 + l''^2 + m''^2 + n''^2}\,,\end{aligned}$$

then we have for the "direction cosines" of the line on $F = 0$ corresponding to $d\zeta$

$$\frac{k''}{\Omega''},\quad -\frac{l''}{\Omega''},\quad \frac{m''}{\Omega''},\quad -\frac{n''}{\Omega''}.$$

In like manner if η alone vary we have, for the "direction cosines" of the line on F corresponding to $d\eta$,

$$\frac{k'}{\Omega'},\quad -\frac{l'}{\Omega'},\quad \frac{m'}{\Omega'},\quad -\frac{n'}{\Omega'},$$

and, finally, for ξ the only varying quantity

$$\frac{k}{\Omega}, \quad -\frac{l}{\Omega}, \quad \frac{m}{\Omega}, \quad -\frac{n}{\Omega}.$$

The conditions for orthogonality are now

$$\begin{aligned} kk' + ll' + mm' + nn' &= 0, \\ k'k'' + l'l'' + m'm'' + n'n'' &= 0, \\ k''k + l''l + m''m + n''n &= 0. \end{aligned}$$

The ratio of the element $d\xi$ to the corresponding element on F is

$$\frac{d\xi}{\sqrt{dx^2 + dy^2 + dz^2 + dt^2}},$$

or,

$$\frac{adx + bdy + cdz + edt}{\sqrt{dx^2 + dy^2 + dz^2 + dt^2}},$$

or finally,

$$\frac{ak - bl + cm - en}{\Omega}.$$

Equating this to the ratio of $d\eta$ to its corresponding element on F,

$$\frac{1}{\Omega}(ak - bl + cm - en) = \frac{1}{\Omega'}(a'k' - b'l' + c'm' - e'n').$$

The quantities in the parentheses are respectively $= \nabla$ and $- \nabla$, so that this equation becomes simply

$$\frac{1}{\Omega} - \frac{1}{\Omega'} = 0,$$

or,

$$\Omega' - \Omega = 0,$$

and in like manner

$$\begin{aligned} \Omega'' - \Omega' &= 0, \\ \Omega - \Omega'' &= 0. \end{aligned}$$

These equations, written out in full, constitute, with the previously given equations of orthogonality, the entire system of equations of condition from which we must endeavor to determine the differential equations affording the solution of the problem. For convenience I give the entire set of equations here; they are:

1.) $kk' + ll' + mm' + nn' = 0,$

2.) $k'k'' + l'l'' + m'm'' + n'n'' = 0,$

3.) $k''k + l''l + m''m + n''n = 0,$

4.) $k'^2 + l'^2 + m'^2 + n'^2 - (k^2 + l^2 + m^2 + n^2) = 0,$

5.) $k''^2 + l''^2 + m''^2 + n''^2 - (k'^2 + l'^2 + m'^2 + n'^2) = 0,$

6.) $k^2 + l^2 + m^2 + n^2 - (k''^2 + l''^2 + m''^2 + n''^2) = 0.$

Multiply the first, second and third of these equations by $2i$ and add and substract the results from 4, 5 and 6 respectively.

7. $(k' \pm ik)^2 + (l' \pm il)^2 + (m' \pm im)^2 + (n' \pm in)^2 = 0,$

8. $(k'' \pm ik')^2 + (l'' \pm il')^2 + (m'' \pm im')^2 + (n'' \pm in')^2 = 0,$

9. $(k \pm ik'')^2 + (l \pm il'')^2 + (m \pm im'')^2 + (n \pm in'')^2 = 0.$

The formation of these expressions from the determinant ∇ is not difficult though a little tedious, so I merely give the results in which, however, for brevity, I have written

$$u_1 = \xi + i\eta, \qquad v_1 = \eta + i\zeta, \qquad w_1 = \zeta + i\xi,$$
$$u_2 = \xi - i\eta, \qquad v_2 = \eta - i\zeta, \qquad w_2 = \zeta - i\xi,$$

and also used the symbols $u_{1,2}$, $v_{1,2}$, $w_{1,2}$, when no confusion could arise, to denote that the expressions are the same in form for u_2 as for u_1, &c. Equation 7 becomes now

$$\begin{vmatrix} \frac{\partial F}{\partial y}, & \frac{\partial F}{\partial z}, & \frac{\partial F}{\partial t} \\ \frac{\partial u_{1,2}}{\partial y}, & \frac{\partial u_{1,2}}{\partial z}, & \frac{\partial u_{1,2}}{\partial t} \\ \frac{\partial \zeta}{\partial y}, & \frac{\partial \zeta}{\partial z}, & \frac{\partial \zeta}{\partial t} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial F}{\partial z}, & \frac{\partial F}{\partial t}, & \frac{\partial F}{\partial x} \\ \frac{\partial u_{1,2}}{\partial z}, & \frac{\partial u_{1,2}}{\partial t}, & \frac{\partial u_{1,2}}{\partial x} \\ \frac{\partial \zeta}{\partial z}, & \frac{\partial \zeta}{\partial t}, & \frac{\partial \zeta}{\partial x} \end{vmatrix}^2$$
$$+ \begin{vmatrix} \frac{\partial F}{\partial t}, & \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y} \\ \frac{\partial u_{1,2}}{\partial t}, & \frac{\partial u_{1,2}}{\partial x}, & \frac{\partial u_{1,2}}{\partial y} \\ \frac{\partial \zeta}{\partial t}, & \frac{\partial \zeta}{\partial x}, & \frac{\partial \zeta}{\partial y} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y}, & \frac{\partial F}{\partial z} \\ \frac{\partial u_{1,2}}{\partial x}, & \frac{\partial u_{1,2}}{\partial y}, & \frac{\partial u_{1,2}}{\partial z} \\ \frac{\partial \zeta}{\partial x}, & \frac{\partial \zeta}{\partial y}, & \frac{\partial \zeta}{\partial z} \end{vmatrix}^2 = 0.$$

The expanded forms of 8 and 9 are obtained from this by merely changing u into v and w successively. Denoting by θ_1, θ_2, θ_3, θ_4 any four quantities whatever and writing,

$$\begin{vmatrix} \theta_1, & \theta_2, & \theta_3, & \theta_4 \\ \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y}, & \frac{\partial F}{\partial z}, & \frac{\partial F}{\partial t} \\ \frac{\partial u_{1,2}}{\partial x}, & \frac{\partial u_{1,2}}{\partial y}, & \frac{\partial u_{1,2}}{\partial z}, & \frac{\partial u_{1,2}}{\partial t} \\ \frac{\partial \zeta}{\partial x}, & \frac{\partial \zeta}{\partial y}, & \frac{\partial \zeta}{\partial z}, & \frac{\partial \zeta}{\partial t} \end{vmatrix} = U_{1,2},$$

we can place the last equation in the form

$$\left(\frac{\partial U_{1,2}}{\partial \theta_1}\right)^2 + \left(\frac{\partial U_{1,2}}{\partial \theta_2}\right)^2 + \left(\frac{\partial U_{1,2}}{\partial \theta_3}\right)^2 + \left(\frac{\partial U_{1,2}}{\partial \theta_4}\right)^2 = 0,$$

or, for convenience, making

$$\left(\frac{\partial}{\partial\theta_1}\right)^2+\left(\frac{\partial}{\partial\theta_2}\right)^2+\left(\frac{\partial}{\partial\theta_3}\right)^2+\left(\frac{\partial}{\partial\theta_4}\right)^2\equiv 0,$$

$\Big($observe that $\left(\frac{\partial}{\partial\theta_i}\right)^2$ is not $=\frac{\partial^2}{\partial\theta_i^2}\Big)$,

$$DU_{1,2}=0,\quad DV_{1,2}=0,\quad DW_{1,2}=0,$$

V and W being quantities of the same kind as U.

The calculation of the quantities $\left(\frac{\partial U_{1,2}}{\partial\theta_i}\right)^2$ is quite simple, since each of them is a symmetrical determinant. If we expand these expressions so that $DU_{1,2}$ shall appear in the form

$$A\alpha^2+B\beta^2+C\gamma^2+D\varepsilon^2-2E\alpha\beta-2F\beta\gamma,\ \text{\&c.}=0,$$

$\alpha, \beta, \gamma, \varepsilon$ being as before the derivatives of F with respect to x, y, z, t, we shall find A, B, C, D given as sums of squares of three determinants of the second order—or, consequently, as one symmetrical determinant of that order; E, F, &c. as sums of products of determinants of the second order. But the most convenient form in which to have the equation is that of one symmetrical determinant of the third order, and by a known theorem we can write it at once as

$$DU_{1,2}=\begin{vmatrix} \alpha^2+\beta^2+\gamma^2+\varepsilon^2, & a_1\alpha+b_1\beta+c_1\gamma+e_1\varepsilon, & a_2\alpha+b_2\beta+c_2\gamma+e_2\varepsilon \\ a_1\alpha+b_1\beta+c_1\gamma+e_1\varepsilon, & a_1^2+b_1^2+c_1^2+e_1^2, & a_1a_2+b_1b_2+c_1c_2+e_1e_2 \\ a_2\alpha+b_2\beta+c_2\gamma+e_2\varepsilon, & a_1a_2+b_1b_2+c_1c_2+e_1e_2, & a_2^2+b_2^2+c_2^2+e_2^2 \end{vmatrix}=0,$$

where a_1, b_1, c_1, e_1 are the derivatives of $u_{1,2}$ with respect to x, y, z, t respectively, a_2, b_2, c_2, e_2 are the same derivatives of ζ. There are two other equations of this form for $v_{1,2}$ and $w_{1,2}$ in which $\alpha, \beta, \gamma, \varepsilon$ have the same significance as in $DU_{1,2}=0$, a_1, b_1, c_1, e_1 are the derivatives however of $v_{1,2}$ and $w_{1,2}$, respectively, and a_2, b_2, c_2, e_2 denote in $DV_{1,2}=0$ the derivatives of ξ and in $DW_{1,2}=0$ the derivatives of η with respect to x, y, z and t.

If we had started to project the surface $F(x, y, z)=0$ in Euclidean space upon the plane ξ, η, we would have found as the equations of the problem

$$\begin{vmatrix} \alpha^2+\beta^2+\gamma^2, & a_1\alpha+b_1\beta+c_1\gamma \\ a_1\alpha+b_1\beta+c_1\gamma, & a_1^2+b_1^2+c_1^2 \end{vmatrix}=0,$$

the α, β, γ having the same meaning, as before, and a_1, b_1, c_1 to take the values

$$\frac{\partial u_1}{\partial x},\quad \frac{\partial u_1}{\partial y},\quad \frac{\partial u_1}{\partial z},$$

$$\frac{\partial u_2}{\partial x},\quad \frac{\partial u_2}{\partial y},\quad \frac{\partial u_2}{\partial z},$$

successively. These are the equations for the orthomorphic projection of a surface upon a plane.

If we have n variables $x_1, x_2, \ldots x_n$ connected by the relation

$$F(x_1, x_2, \ldots x_n) = 0,$$

the general locus of n-dimensional space, and $n-1$ variables $\xi_1, \xi_2, \ldots \xi_{n-1}$ denoting the coordinates in space of $n-1$ dimensions, then for the projection of $F=0$ into the $n-1$-dimensional space, we shall have $\frac{(n-1)(n-2)}{2}$ groups of equations of the form

$$\text{A.}\quad \left\{ \begin{array}{l} \left(\frac{\partial U_{1,2}}{\partial \theta_1}\right)^2 + \left(\frac{\partial U_{1,2}}{\partial \theta_2}\right)^2 + \ldots \left(\frac{\partial U_{1,2}}{\partial \theta_n}\right)^2 = 0, \\ \left(\frac{\partial V_{1,2}}{\partial \theta_1}\right)^2 + \left(\frac{\partial V_{1,2}}{\partial \theta_2}\right)^2 + \ldots \left(\frac{\partial V_{1,2}}{\partial \theta_n}\right)^2 = 0, \end{array} \right.$$

the quantities θ being anything whatever, and the derivatives

$$\frac{\partial U}{\partial \theta}, \quad \frac{\partial V}{\partial \theta},$$

being the first minors of the determinants

$$\left| \begin{array}{llll} \theta_1\ , & \theta_2\ , & \ldots & \theta_n \\ \frac{\partial F}{\partial x_1}, & \frac{\partial F}{\partial x_2}, & \ldots & \frac{\partial F}{\partial x_n} \\ \frac{\partial u_{1,2}}{\partial x_1}, & \frac{\partial u_{1,2}}{\partial x_2}, & \ldots & \frac{\partial u_{1,2}}{\partial x_n} \\ \frac{\partial \xi_3}{\partial x_1}, & \frac{\partial \xi_3}{\partial x_2}, & \ldots & \frac{\partial \xi_3}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \xi_{n-1}}{\partial x_1}, & \frac{\partial \xi_{n-1}}{\partial x_1}, & \ldots & \frac{\partial \xi_{n-1}}{\partial x_n} \end{array} \right| = V_{1,2}$$

and

$$\left| \begin{array}{llll} \theta_1\ , & \theta_2\ , & \ldots & \theta_n \\ \frac{\partial F}{\partial x_1}, & \frac{\partial F}{\partial x_2}, & \ldots & \frac{\partial F}{\partial x_n} \\ \frac{\partial v_{1,2}}{\partial x_1}, & \frac{\partial v_{1,2}}{\partial x_2}, & \ldots & \frac{\partial v_{1,2}}{\partial x_n} \\ \frac{\partial \xi_3}{\partial x_1}, & \frac{\partial \xi_3}{\partial x_2}, & \ldots & \frac{\partial \xi_3}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \xi_{n-1}}{\partial x_1}, & \frac{\partial \xi_{n-1}}{\partial x_2}, & \ldots & \frac{\partial \xi_{n-1}}{\partial x_n} \end{array} \right| = V_{1,2}$$

where

$$u_{1,2}=\xi_1+i\xi_2,\quad u_{2,3}=\xi_2+i\xi_3,\ \dots\ u_{i,i+1}=\xi_i+i\xi_{i+1},$$
$$v_{1,2}=\xi_1-i\xi_2,\quad v_{2,3}=\xi_2-i\xi_3,\ \dots\ v_{i,i+1}=\xi_i-i\xi_{i+1}.$$

The remaining $\frac{n(n-3)}{2}$ groups of equations similar to A are of course formed from this group by changing the $u_{1,2}$ into $u_{2,3}$, $u_{3,4}$, &c., and $v_{1,2}$ into $v_{2,3}$, $v_{3,4}$, &c., and also changing the ξ by advancing successively all the subscripts $1, 2 \dots n-1$.

By a theorem of determinants (which was unknown to me until kindly communicated by Mr. Stringham) each of equations A may be put in the following form:

$$\begin{vmatrix}
\Sigma\left(\frac{\partial F}{\partial x_i}\right)^2, & \Sigma\frac{\partial F}{\partial x_i}\frac{\partial u_{1,2}}{\partial x_i}, & \Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_3}{\partial x_i}, & \dots\dots\dots\dots & \Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i} \\
\Sigma\frac{\partial F}{\partial x_i}\frac{\partial u_{1,2}}{\partial x_i}, & \Sigma\left(\frac{\partial u_{1,2}}{\partial x_i}\right)^2, & \Sigma\frac{\partial u_{1,2}}{\partial x_i}\frac{\partial \xi_3}{\partial x_i}, & \dots\dots\dots\dots & \Sigma\frac{\partial u_{1,3}}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i} \\
\Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_3}{\partial x_i}, & \Sigma\frac{\partial u_{1,2}}{\partial x_i}\frac{\partial \xi_3}{\partial x_i}, & \Sigma\left(\frac{\partial \xi_3}{\partial x_i}\right)^2, & \dots\dots\dots\dots & \Sigma\frac{\partial \xi_3}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i} \\
\Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_4}{\partial x_i}, & \Sigma\frac{\partial u_{1,2}}{\partial x_i}\frac{\partial \xi_4}{\partial x_i}, & \Sigma\frac{\partial \xi_3}{\partial x_i}\frac{\partial \xi_4}{\partial x_i}, & \Sigma\left(\frac{\partial \xi_4}{\partial x_i}\right)^2, \dots & \Sigma\frac{\partial \xi_4}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i} \\
\dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots \\
\Sigma\frac{\partial F}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i}, & \Sigma\frac{\partial u_{1,2}}{\partial x_i}\frac{\partial \xi_{n-1}}{\partial x_i}, & \dots\dots\dots\dots & \dots\dots\dots\dots & \Sigma\left(\frac{\partial \xi_{n-1}}{\partial x_i}\right)^2
\end{vmatrix}=0,$$

the summations to be taken from $i=1$ to $i=n$. The equation for V being similar to this, there is no necessity for writing it.

WASHINGTON, *Sept. 4, 1879.*